PROBABILITY FOR RISK MANAGEMENT

by

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Second Edition

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Preface
to the Second Edition

The major change in this new edition is an increase in the number of challenging problems. This was requested by our readers. Since the actuarial examinations are an excellent source of challenging problems, we have added 109 sample exam problems to our exercise sections. (Detailed solutions can be found in the solutions manual). We thank the Society of Actuaries for permission to use these problems.

We have added three new sections which cover the bivariate normal distribution, joint moment generating functions and the multinomial distribution.

The authors would like to thank the second edition review team: Leonard A. Asimow, ASA, Ph.D. Robert Morris University, and Krupa S. Viswanathan, ASA, Ph.D., Temple University.

Finally we would like to thank Gail Hall for her editorial work on the text and Marilyn Baleshiski for putting the book together.

Matt Hassett
Don Stewart

Tempe, Arizona
June, 2006

Preface

This text provides a first course in probability for students with a basic calculus background. It has been designed for students who are mostly interested in the applications of probability to risk management in vital modern areas such as insurance, finance, economics, and health sciences. The text has many features which are tailored for those students.

Integration of applications and theory. Much of modern probability theory was developed for the analysis of important risk management problems. The student will see here that each concept or technique applies not only to the standard card or dice problems, but also to the analysis of insurance premiums, unemployment durations, and lives of mortgages. Applications are not separated as if they were an afterthought to the theory. The concept of pure premium for an insurance is introduced in a section on expected value because the pure premium is an expected value.

Relevant applications. Applications will be taken from texts, published studies, and practical experience in actuarial science, finance, and economics.

Development of key ideas through well-chosen examples. The text is not abstract, axiomatic or proof-oriented. Rather, it shows the student how to use probability theory to solve practical problems. The student will be introduced to Bayes' Theorem with practical examples using trees and then shown the relevant formula. Expected values of distributions such as the gamma will be presented as useful facts, with proof left as an honors exercise. The student will focus on applying Bayes' Theorem to disease testing or using the gamma distribution to model claim severity.

Emphasis on intuitive understanding. Lack of formal proofs does not correspond to a lack of basic understanding. A well-chosen tree example shows most students what Bayes' Theorem is really doing. A simple expected value calculation for the exponential distribution or a polynomial density function demonstrates how expectations are found. The student should feel that he or she understands each concept. The words “beyond the scope of this text” will be avoided.

Organization as a useful future reference. The text will present key formulas and concepts in clearly identified formula boxes and provide useful summary tables. For example, Appendix B will list all major distributions covered, along with the density function, mean, variance, and moment generating function of each.

Use of technology. Modern technology now enables most students to solve practical problems which were once thought to be too involved. Thus students might once have integrated to calculate probabilities for an exponential distribution, but avoided the same problem for a gamma distribution with $\alpha = 5$ and $\beta = 3$. Today any student with a TI-83 calculator or a personal computer version of MATLAB or Maple or Mathematica can calculate probabilities for the latter distribution. The text will contain boxed Technology Notes which show what can be done with modern calculating tools. These sections can be omitted by students or teachers who do not have access to this technology, or required for classes in which the technology is available.

The practical and intuitive style of the text will make it useful for a number of different course objectives.

A first course in probability for undergraduate mathematics majors. This course would enable sophomores to see the power and excitement of applied probability early in their programs, and provide an incentive to take further probability courses at higher levels. It would be especially useful for mathematics majors who are considering careers in actuarial science.

An incentive course for talented business majors. The probability methods contained here are used on Wall Street, but they are not generally required of business students. There is a large untapped pool of mathematically-talented business students who could use this course experience as a base for a career as a “rocket scientist” in finance or as a mathematical economist.

An applied review course for theoretically-oriented students. Many mathematics majors in the United States take only an advanced, proof-oriented course in probability. This text can be used for a review of basic material in an understandable applied context. Such a review may be particularly helpful to mathematics students who decide late in their programs to focus on actuarial careers.

The text has been class-tested twice at Arizona State University. Each class had a mixed group of actuarial students, mathematically- talented students from other areas such as economics, and interested mathematics majors. The material covered in one semester was Chapters 1-7, Sections 8.1-8.5, Sections 9.1-9.4, Chapter 10 and Sections 11.1-11.4. The text is also suitable for a pre-calculus introduction to probability using Chapters 1-6, or a two-semester course which covers the entire text. As always, the amount of material covered will depend heavily on the preferences of the instructor.

The authors would like to thank the following members of a review team which worked carefully through two draft versions of this text:

Sam Broverman, ASA, Ph.D., University of Toronto
Sheldon Eisenberg, Ph.D., University of Hartford
Bryan Hearsey, ASA, Ph.D., Lebanon Valley College
Tom Herzog, ASA, Ph.D., Department of HUD
Eugene Spiegel, Ph.D., University of Connecticut

The review team made many valuable suggestions for improvement and corrected many errors. Any errors which remain are the responsibility of the authors.

A second group of actuaries reviewed the text from the point of view of the actuary working in industry. We would like to thank William Gundberg, EA, Brian Januzik, ASA, and Andy Ribaudo, ASA, ACAS, FCAS, for valuable discussions on the relation of the text material to the day-to-day work of actuarial science.

Special thanks are due to others. Dr. Neil Weiss of Arizona State University was always available for extremely helpful discussions concerning subtle technical issues. Dr. Michael Ratliff, ASA, of Northern Arizona University and Dr. Stuart Klugman, FSA, of Drake University read the entire text and made extremely helpful suggestions.

Thanks are also due to family members. Peggy Craig-Hassett provided warm and caring support throughout the entire process of creating this text. John, Thia, Breanna, JJ, Laini, Ben, Flint, Elle and Sabrina all enriched our lives, and also provided motivation for some of our examples.

We would like to thank the ACTEX team which turned the idea for this text into a published work. Richard (Dick) London, FSA, first proposed the creation of this text to the authors and has provided editorial guidance through every step of the project. Denise Rosengrant did the daily work of turning our copy into an actual book.

Finally a word of thanks for our students. Thank you for working with us through two semesters of class-testing, and thank you for your positive and cooperative spirit throughout. In the end, this text is not ours. It is yours because it will only achieve its goals if it works for you.

May, 1999 Matthew J. Hassett
Tempe, Arizona Donald G. Stewart

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Ty and Jake,

Flint,

Xochil

Chapter 1
Probability: A Tool for Risk Management

1.1 Who Uses Probability?

Probability theory is used for decision-making and risk management throughout modern civilization. Individuals use probability daily, whether or not they know the mathematical theory in this text. If a weather forecaster says that there is a 90% chance of rain, people carry umbrellas. The "90% chance of rain" is a statement of a probability. If a doctor tells a patient that a surgery has a 50% chance of an unpleasant side effect, the patient may want to look at other possible forms of treatment. If a famous stock market analyst states that there is a 90% chance of a severe drop in the stock market, people sell stocks. All of us make decisions about the weather, our finances and our health based on percentage statements which are really probability statements.

Because probabilities are so important in our analysis of risk, professionals in a wide range of specialties study probability. Weather experts use probability to derive the percentages given in their forecasts. Medical researchers use probability theory in their study of the effectiveness of new drugs and surgeries. Wall Street firms hire mathematicians to apply probability in the study of investments.

The insurance industry has a long tradition of using probability to manage its risks. If you want to buy car insurance, the price you will pay is based on the probability that you will have an accident. (This price is called a **premium**.) Life insurance becomes more expensive to purchase as you get older, because there is a higher probability that you will die. Group health insurance rates are based on the study of the probability that the group will have a certain level of claims.

The professionals who are responsible for the risk management and premium calculation in insurance companies are called actuaries. Actuaries take a long series of exams to be certified, and those exams emphasize mathematical probability because of its importance in insurance risk management. Probability is also used extensively in investment analysis, banking and corporate finance. To illustrate the application of probability in financial risk management, the next section gives a simplified example of how an insurance rate might be set using probabilities.

1.2 An Example from Insurance

In 2002 deaths from motor vehicle accidents occurred at a rate of 15.5 per 100,000 population.[1] This is really a statement of a probability. A mathematician would say that the probability of death from a motor vehicle accident in the next year is $15.5/100{,}000 = .000155$.

Suppose that you decide to sell insurance and offer to pay \$10,000 if an insured person dies in a motor vehicle accident. (The money will go to a beneficiary who is named in the policy — perhaps a spouse, a close friend, or the actuarial program at your alma mater.) Your idea is to charge for the insurance and use the money obtained to pay off any claims that may occur. The tricky question is what to charge.

You are optimistic and plan to sell 1,000,000 policies. If you believe the rate of 15.5 deaths from motor vehicles per 100,000 population still holds today, you would expect to have to pay 155 claims on your 1,000,000 policies. You will need $155(10{,}000) = \$1{,}550{,}000$ to pay those claims. Since you have 1,000,000 policyholders, you can charge each one a premium of \$1.55. The charge is small, but $1.55(1{,}000{,}000) = \$1{,}550{,}000$ gives you the money you will need to pay claims.

This example is oversimplified. In the real insurance business you would earn interest on the premiums until the claims had to be paid. There are other more serious questions. Should you expect exactly 155 claims from your 1,000,000 clients just because the national rate is 15.5 claims in 100,000? Does the 2002 rate still apply? How can you pay expenses and make a profit in addition to paying claims? To answer these questions requires more knowledge of probability, and that is why

[1] *Statistical Abstract of the United States*, 1996. Table No. 138, page 101.

this text does not end here. However, the oversimplified example makes a point. Knowledge of probability can be used to pool risks and provide useful goods like insurance. The remainder of this text will be devoted to teaching the basics of probability to students who wish to apply it in areas such as insurance, investments, finance and medicine.

1.3 Probability and Statistics

Statistics is a discipline which is based on probability but goes beyond probability to solve problems involving inferences based on sample data. For example, statisticians are responsible for the opinion polls which appear almost every day in the news. In such polls, a sample of a few thousand voters are asked to answer a question such as "Do you think the president is doing a good job?" The results of this sample survey are used to make an inference about the percentage of all voters who think that the president is doing a good job. The insurance problem in Section 1.2 requires use of both probability and statistics. In this text, we will not attempt to teach statistical methods, but we will discuss a great deal of probability theory that is useful in statistics. It is best to defer a detailed discussion of the difference between probability and statistics until the student has studied both areas. It is useful to keep in mind that the disciplines of probability and statistics are related, but not exactly the same.

1.4 Some History

The origins of probability are a piece of everyday life; the subject was developed by people who wished to gamble intelligently. Although games of chance have been played for thousands of years, the development of a systematic mathematics of probability is more recent. Mathematical treatments of probability appear to have begun in Italy in the latter part of the fifteenth century. A gambler's manual which considered interesting problems in probability was written by Cardano (1500-1572).

The major advance which led to the modern science of probability was the work of the French mathematician Blaise Pascal. In 1654 Pascal was given a gaming problem by the gambler Chevalier de Mere. The problem of points dealt with the division of proceeds of an interrupted game. Pascal entered into correspondence with another French mathematician, Pierre de Fermat. The problem was solved in this correspondence, and this work is regarded as the starting point for modern probability.

It is important to note that within twenty years of Pascal's work, differential and integral calculus was being developed (independently) by Newton and Leibniz. The subsequent development of probability theory relied heavily on calculus.

Probability theory developed at a steady pace during the eighteenth and nineteenth centuries. Contributions were made by leading scientists such as James Bernoulli, de Moivre, Legendre, Gauss and Poisson. Their contributions paved the way for very rapid growth in the twentieth century.

Probability is of more recent origin than most of the mathematics covered in university courses. The computational methods of freshman calculus were known in the early 1700's, but many of the probability distributions in this text were not studied until the 1900's. The applications of probability in risk management are even more recent. For example, the foundations of modern portfolio theory were developed by Harry Markowitz [11] in 1952. The probabilistic study of mortgage prepayments was developed in the late 1980's to study financial instruments which were first created in the 1970's and early 1980's.

It would appear that actuaries have a longer tradition of use of probability; a text on life contingencies was published in 1771.[2] However, modern stochastic probability models did not seriously influence the actuarial profession until the 1970's, and actuarial researchers are now actively working with the new methods developed for use in modern finance. The July 2005 copy of the North American Actuarial Journal that is sitting on my desk has articles with titles like "Minimizing the Probability of Ruin When Claims Follow Brownian Motion With Drift." You can't read this article unless you know the basics contained in this book and some more advanced topics in probability.

Probability is a young area, with most of its growth in the twentieth century. It is still developing rapidly and being applied in a wide range of practical areas. The history is of interest, but the future will be much more interesting.

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2 See the section on Historical Background in the 1999 Society of Actuaries Yearbook, page 5.

1.5 Computing Technology

Modern computing technology has made some practical problems easier to solve. Many probability calculations involve rather difficult integrals; we can now compute these numerically using computers or modern calculators. Some problems are difficult to solve analytically but can be studied using computer simulation. In this text we will give examples of the use of technology in most sections. We will refer to results obtained using the TI-83 and TI BA II Plus Professional calculators and Microsoft® EXCEL. but will not attempt to teach the use of those tools. The technology sections will be clearly boxed off to separate them from the remainder of the text. Students who do not have the technological background should be aware that this will in no way restrict their understanding of the theory. However, the technology discussions should be valuable to the many students who already use modern calculators or computer packages.

Chapter 2
Counting for Probability

2.1 What Is Probability?

People who have never studied the subject understand the intuitive ideas behind the mathematical concept of probability. Teachers (including the authors of this text) usually begin a probability course by asking the students if they know the probability of a coin toss coming up heads. The obvious answer is 50% or ½, and most people give the obvious answer with very little hesitation. The reasoning behind this answer is simple. There are two possible outcomes of the coin toss, heads or tails. If the coin comes up heads, only one of the two possible outcomes has occurred. There is one chance in two of tossing a head.

The simple reasoning here is based on an assumption — *the coin must be fair, so that heads and tails are **equally likely***. If your gambler friend Fast Eddie invites you into a coin tossing game, you might suspect that he has altered the coin so that he can get your money. However, if you are willing to assume that the coin is fair, you count possibilities and come up with ½.

Probabilities are evaluated by counting in a wide variety of situations. Gambling related problems involving dice and cards are typically solved using counting. For example, suppose you are rolling a single six-sided die whose sides bear the numbers 1, 2, 3, 4, 5 and 6. You wish to bet on the event that you will roll a number less than 5. The probability of this event is 4/6, since the outcomes $1, 2, 3$ and 4 are less than 5 and there are six possible outcomes (assumed equally likely). The approach to probability used is summarized as follows:

Probability by Counting for Equally Likely Outcomes

$$\textit{Probability of an event} = \frac{\textit{Number of outcomes in the event}}{\textit{Total number of possible outcomes}}$$

Part of the work of this chapter will be to introduce a more precise mathematical framework for this counting definition. However, this is not the only way to view probability. There are some cases in which outcomes may not be equally likely. A die or a coin may be altered so that all outcomes are not equally likely. Suppose that you are tossing a coin and suspect that it is not fair. Then the probability of tossing a head cannot be determined by counting, but there is a simple way to *estimate* that probability — simply toss the coin a large number of times and count the number of heads. If you toss the coin 1000 times and observe 650 heads, your best estimate of the probability of a head on one toss is 650/1000 = .65. In this case you are using a **relative frequency estimate** of a probability.

Relative Frequency Estimate of the Probability of an Event

$$\textit{Probability of an event} = \frac{\textit{Number of times the event occurs in } n \textit{ trials}}{n}$$

We now have two ways of looking at probability, the counting approach for equally likely outcomes and the relative frequency approach. This raises an interesting question. If outcomes are equally likely, will both approaches lead to the same probability? For example, if you try to find the probability of tossing a head for a fair coin by tossing the coin a large number of times, should you expect to get a value of ½? The answer to this question is “not exactly, but for a very large number of tosses you are highly likely to get an answer close to ½.” The more tosses, the more likely you are to be very close to ½. We had our computer simulate different numbers of coin tosses, and came up with the following results.

Number of Tosses	Number of Heads	Probability Estimate
4	1	.25
100	54	.54
1000	524	.524
10,000	4985	.4985

More will be said later in the text about the mathematical reasoning underlying the relative frequency approach. Many texts identify a third approach to probability. That is the **subjective** approach to probability. Using this approach, you ask a well-informed person for his or her personal estimate of the probability of an event. For example, one of your authors worked on a business valuation problem which required knowledge of the probability that an individual would fail to make a monthly mortgage payment to a company. He went to an executive of the company and asked what percent of individuals failed to make the monthly payment in a typical month. The executive, relying on his experience, gave an estimate of 3%, and the valuation problem was solved using a subjective probability of .03. The executive's subjective estimate of 3% was based on a personal recollection of relative frequencies he had seen in the past.

In the remainder of this chapter we will work on building a more precise mathematical framework for probability. The counting approach will play a big part in this framework, but the reader should keep in mind that many of the probability numbers actually used in calculation may come from relative frequencies or subjective estimates.

2.2 The Language of Probability; Sets, Sample Spaces and Events

If probabilities are to be evaluated by counting outcomes of a probability experiment, it is essential that all outcomes be specified. A person who is not familiar with dice does not know that the possible outcomes for a single die are 1, 2, 3, 4, 5 and 6. That person cannot find the probability of rolling a 1 with a single die because the basic outcomes are unknown. In every well-defined probability experiment, all possible outcomes must be specified in some way.

The language of set theory is very useful in the analysis of outcomes. Sets are covered in most modern mathematics courses, and the reader is assumed to be familiar with some set theory. For the sake of completeness, we will review some of the basic ideas of set theory. A **set** is a collection of objects such as the numbers $1, 2, 3, 4, 5$ and 6. These objects are called the **elements** or **members** of the set. If the set is finite and small enough that we can easily list all of its elements, we can describe the set by listing all of its elements in braces. For the set above, $S = \{1, 2, 3, 4, 5, 6\}$. For large or infinite sets, the set-builder notation is helpful. For example, the set of all positive real numbers may be written as

$$S = \{x \mid x \text{ is a real number and } x > 0\}.$$

Often it is assumed that the numbers in question are real numbers, and the set above is written as $S = \{x \mid x > 0\}$.

We will review more set theory as needed in this chapter. The important use of set theory here is to provide a precise language for dealing with the outcomes in a probability experiment. The definition below uses the set concept to refer to all possible outcomes of a probability experiment.

Definition 2.1 The **sample space** S for a probability experiment is the set of all possible outcomes of the experiment.

Example 2.1 A single die is rolled and the number facing up recorded. The sample space is $S = \{1, 2, 3, 4, 5, 6\}$. □

Example 2.2 A coin is tossed and the side facing up is recorded. The sample space is $S = \{H, T\}$. □

Many interesting applications involve a simple two-element sample space. The following examples are of this type.

Example 2.3 (Death of an insured) An insurance company is interested in the probability that an insured will die in the next year. The sample space is $S = \{$*death, survival*$\}$. □

Example 2.4 (Failure of a part in a machine) A manufacturer is interested in the probability that a crucial part in a machine will fail in the next week. The sample space is $S = \{$*failure, survival*$\}$. □

Example 2.5 (Default of a bond) Companies borrow money they need by issuing **bonds**. A bond is typically sold in \$1000 units which have a fixed interest rate such as 8% per year for twenty years. When you buy a bond for \$1000, you are actually loaning the company your \$1000 in return for 8% interest per year. You are supposed to get your \$1000 loan back in twenty years. If the company issuing the bonds has financial trouble, it may declare bankruptcy and *default* by failing to pay your money back. Investors who buy bonds wish to find the probability of default. The sample space is $S = \{\textit{default, no default}\}$. □

Example 2.6 (Prepayment of a mortgage) Homeowners usually buy their homes by getting a **mortgage loan** which is repaid by monthly payments. The homeowner usually has the right to pay off the mortgage loan early if that is desirable — because the homeowner decides to move and sell the house, because interest rates have gone down, or because someone has won the lottery. Lenders may lose or gain money when a loan is prepaid early, so they are interested in the probability of prepayment. If the lender is interested in whether the loan will prepay in the next month, the sample space is $S = \{\textit{prepayment, no prepayment}\}$. □

The simple sample spaces above are all of the same type. Something (a bond, a mortgage, a person, or a part) either continues or disappears. Despite this deceptive simplicity, the probabilities involved are of great importance. If a part in your airplane fails, you may become an insurance death — leading to the prepayment of your mortgage and a strain on your insurance company and its bonds. The probabilities are difficult and costly to estimate. Note also that the coin toss sample space $\{H, T\}$ was the only one in which the two outcomes were equally likely. Luckily for most of us, insured individuals are more likely to live than die and bonds are more likely to succeed than to default.

Not all sample spaces are so small or so simple.

Example 2.7 An insurance company has sold 100 individual life insurance policies. When an insured individual dies, the beneficiary named in the policy will file a claim for the amount of the policy. You wish to observe the number of claims filed in the next year. The sample space consists of all integers from 0 to 100, so $S = \{0, 1, 2, \ldots, 100\}$. □

Some of the previous examples may be looked at in slightly different ways that lead to different sample spaces. The sample space is determined by the question you are asking.

Example 2.8 An insurance company sells life insurance to a 30-year-old female. The company is interested in the age of the insured when she eventually dies. If the company assumes that the insured will not live to 110, the sample space is $S = \{30, 31, \ldots, 109\}$. □

Example 2.9 A mortgage lender makes a 30-year monthly payment loan. The lender is interested in studying the month in which the mortgage is paid off. Since there are 360 months in 30 years, the sample space is $S = \{1, 2, 3, ..., 359, 360\}$. □

The sample space can also be infinite.

Example 2.10 A stock is purchased for \$100. You wish to observe the price it can be sold for in one year. Since stock prices are quoted in dollars and fractions of dollars, the stock could have any non-negative rational number as its future value. The sample space consists of all non-negative rational numbers, $S = \{x \mid x \geq 0 \text{ and } x \text{ rational}\}$. This does not imply that the price outcome of \$1,000,000,000 is highly likely in one year — just that it is possible. Note that the price outcome of 0 is also possible. Stocks can become worthless. □

The above examples show that the sample space for an experiment can be a small finite set, a large finite set, or an infinite set.

In Section 2.1 we looked at the probability of events which were specified in words, such as “toss a head” or “roll a number less than 5.” These events also need to be translated into clearly specified sets. For example, if a single die is rolled, the event “roll a number less than 5” consists of the outcomes in the set $E = \{1, 2, 3, 4\}$. Note that the set E is a subset of the sample space S, since every element of E is an element of S. This leads to the following set-theoretical definition of an event.

Definition 2.2 An **event** is a subset of the sample space S.

This set-theoretic definition of an event often causes some unnecessary confusion since people think of an event as something described in words like “roll a number less than 5 on a roll of a single die." There is no conflict here. The definition above reminds you that you must take the event described in words and determine precisely what outcomes are in the event. Below we give a few examples of events which are stated in words and then translated into subsets of the sample space.

Example 2.11 A coin is tossed. You wish to find the probability of the event "toss a head." The sample space is $S = \{H, T\}$. The event is the subset $E = \{H\}$. $\square$

Example 2.12 An insurance company has sold 100 individual life policies. The company is interested in the probability that at most 5 of the policies have death benefit claims in the next year. The sample space is $S = \{0, 1, 2, \ldots, 100\}$. The event E is the subset $\{0, 1, 2, 3, 4, 5\}$. $\square$

Example 2.13 You buy a stock for \$100 and plan to sell it one year later. You are interested in the event E that you make a profit when the stock is sold. The sample space is $S = \{x \mid x \geq 0 \text{ and } x \text{ rational}\}$, the set of all possible future prices. The event E is the subset $E = \{x \mid x > 100 \text{ and } x \text{ rational}\}$, the set of all possible future prices which are greater than the \$100 you paid. $\square$

Problems involving selections from a standard 52 card deck are common in beginning probability courses. Such problems reflect the origins of probability. To make listing simpler in card problems, we will adopt the following abbreviation system:

A: Ace	K: King	Q: Queen	J: Jack
S: Spade	H: Heart	D: Diamond	C: Club

We can then describe individual cards by combining letters and numbers. For example KH will stand for the king of hearts and $2D$ for the 2 of diamonds.

Example 2.14 A standard 52 card deck is shuffled and a card is picked at random. You are interested in the event that the card is a king. The sample space, $S = \{AS, KS, \ldots, 3C, 2C\}$, consists of all 52 cards. The event E consists of the four kings, $E = \{KS, KH, KD, KC\}$. $\square$

The examples of sample spaces and events given above are straightforward. In many practical problems things become much more complex. The following sections introduce more set theory and some counting techniques which will help in analyzing more difficult problems.

2.3 Compound Events; Set Notation

When we refer to events in ordinary language, we often negate them (the card drawn is *not* a king) or combine them using the words "and" or "or" (the card drawn is a king *or* an ace). Set theory has a convenient notation for use with such **compound events**.

2.3.1 Negation

The event *not* E is written as $\sim E$. (This may also be written as $\overline{E}$.)

Example 2.15 A single die is rolled, $S = \{1, 2, 3, 4, 5, 6\}$. The event E is the event of rolling a number less than 5, so $E = \{1, 2, 3, 4\}$. E does *not* occur when a 5 or 6 is rolled. Thus $\sim E = \{5, 6\}$. □

Note that the event $\sim E$ is the set of all outcomes in the sample space which are not in the original event set E. The result of removing all elements of E from the original sample space S is referred to as $S - E$. Thus $\sim E = S - E$. This set is called the **complement** of E.

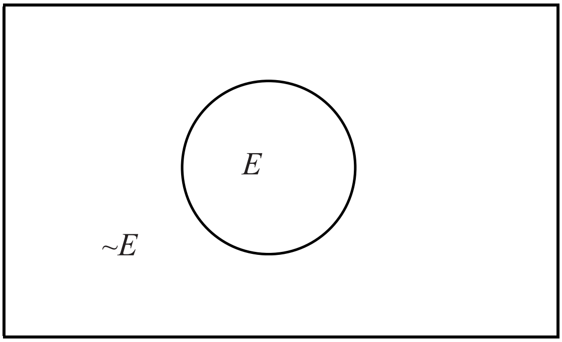
Example 2.16 You buy a stock for \$100 and wish to evaluate the probability of selling it for a higher price x in one year. The sample space is $S = \{x | x \geq 0$ and x rational$\}$. The event of interest is $E = \{x \mid x > 100$ and x rational$\}$. The negation $\sim E$ is the event that no profit is made on the sale, so $\sim E$ can be written as

$$\sim E = \{x \mid 0 \leq x \leq 100 \text{ and } x \text{ rational}\} = S - E.$$

This can be portrayed graphically on a number line.

□

Graphical depiction of events is very helpful. The most common tool for this is the **Venn diagram**, in which the sample space is portrayed as a rectangular region and the event is portrayed as a circular region inside the rectangle. The Venn diagram showing E and ~E is given in the following figure.



2.3.2 The Compound Events *A or B*, *A and B*

We will begin by returning to the familiar example of rolling a single die. Suppose that we have the opportunity to bet on two different events:

A: an even number is rolled	B: a number less than 5 is rolled
$A = \{2, 4, 6\}$	$B = \{1, 2, 3, 4\}$

If we bet that A *or* B occurs, we will win if any element of the two sets above is rolled.

$$A \textit{ or } B = \{1, 2, 3, 4, 6\}$$

In forming the set for A *or* B we have combined the sets A and B by listing all outcomes which appear in *either* A or B. The resulting set is called the **union** of A and B, and is written as $A \cup B$. It should be clear that for any two events A and B

$$A \textit{ or } B = A \cup B.$$

For the single die roll above, we could also decide to bet on the event A *and* B. In that case, *both* the event A and the event B must occur on the single roll. This can happen only if an outcome occurs which is common to both events.

$$A \textit{ and } B = \{2, 4\}$$

In forming the set for A *and* B we have listed all outcomes which are in both sets simultaneously. This set is referred to as the **intersection** of A and B, and is written as $A \cap B$. For any two events A and B

$$A \textit{ and } B = A \cap B.$$

Example 2.17 Consider the insurance company which has written 100 individual life insurance policies and is interested in the number of claims which will occur in the next year. The sample space is $S = \{0, 1, 2, \ldots, 100\}$. The company is interested in the following two events:

A: there are at most 8 claims
B: the number of claims is between 5 and 12 (inclusive)

A and B are given by the sets

$$A = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$$

and

$$B = \{5, 6, 7, 8, 9, 10, 11, 12\}.$$

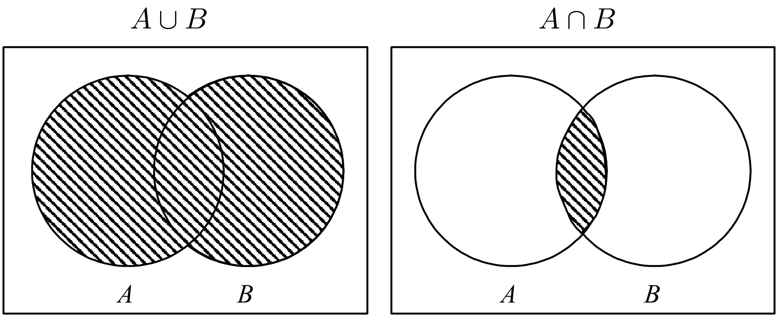
Then the events A *or* B and A *and* B are given by

$$A \textit{ or } B = A \cup B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

and

$$A \textit{ and } B = A \cap B = \{5, 6, 7, 8\}. \qquad \square$$

The events A *or* B and A *and* B can also be represented using Venn diagrams, with overlapping circular regions representing A and B.



2.3.3 New Sample Spaces from Old; Ordered Pair Outcomes

In some situations the basic outcomes of interest are actually pairs of simpler outcomes. The following examples illustrate this.

Example 2.18 (Insurance of a couple) Sometimes life insurance is written on a husband and wife. Suppose the insurer is interested in whether one or both members of the couple die in the next year. Then the insurance company must start by considering the following outcomes:

D_H: death of the husband	S_H: survival of the husband
D_W: death of the wife	S_W: survival of the wife

Since the insurance company has written a policy insuring both husband and wife, the sample space of interest consists of pairs which show the status of both husband and wife. For example, the pair (D_H, S_W) describes the outcome in which the husband dies but the wife survives. The sample space is

$$S = \{(D_H, S_W), (D_H, D_W), (S_H, S_W), (S_H, D_W)\}.$$

In this sample space, events may be more complicated than they sound. Consider the following event:

H: the husband dies in the next year

$$H = \{(D_H, S_W), (D_H, D_W)\}$$

The death of the husband is not a single outcome. The insurance company has insured two people, and has different obligations for each of the two outcomes in H. The death of the wife is similar.

$$W\text{: the wife dies in the next year}$$

$$W = \{(D_H, D_W), (S_H, D_W)\}$$

The events H *or* W and H *and* W are also sets of pairs.

$$H \cup W = \{(D_H, S_W), (D_H, D_W), (S_H, D_W)\}$$

$$H \cap W = \{(D_H, D_W)\} \qquad \square$$

Similar reasoning can be used in the study of the failure of two crucial parts in a machine or the prepayment of two mortgages.

2.4 Set Identities

2.4.1 The Distributive Laws for Sets

The distributive law for real numbers is the familiar

$$a(b + c) = ab + ac.$$

Two similar distributive laws for set operations are the following:

$$\boxed{\begin{aligned} A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \qquad (2.1)\\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \qquad (2.2)\end{aligned}}$$

These laws are helpful in dealing with compound events involving the connectives *and* and *or*. They tell us that

A *and* $(B$ *or* $C)$ <u>is equivalent to</u> $(A$ *and* $B)$ *or* $(A$ *and* $C)$

and

A *or* $(B$ *and* $C)$ <u>is equivalent to</u> $(A$ *or* $B)$ *and* $(A$ *or* $C)$.

The validity of these laws can be seen using Venn diagrams. This is pursued in the exercises. These identities are illustrated in the following example.

Example 2.19 A financial services company is studying a large pool of individuals who are potential clients. The company offers to sell its clients stocks, bonds and life insurance. The events of interest are the following:

S: the individual owns stocks

B: the individual owns bonds

I: the individual has life insurance coverage

The distributive laws tell us that

$$I \cap (B \cup S) = (I \cap B) \cup (I \cap S)$$

and

$$I \cup (B \cap S) = (I \cup B) \cap (I \cup S).$$

The first identity states that

insured *and* (owning bonds *or* stocks)

is equivalent to

(insured *and* owning bonds) *or* (insured *and* owning stocks).

The second identity states that

insured *or* (owning bonds *and* stocks)

is equivalent to

(insured *or* owning bonds) *and* (insured *or* owning stocks). □

2.4.2 De Morgan's Laws

Two other useful set identities are the following:

$$\boxed{\begin{aligned} \sim(A \cup B) &= \sim A \cap \sim B && (2.3) \\ \sim(A \cap B) &= \sim A \cup \sim B && (2.4) \end{aligned}}$$

These laws state that

$$not(A \textit{ or } B) \; \underline{\text{is equivalent to}} \; (not\ A) \textit{ and } (not\ B)$$

and

$$not(A \textit{ and } B) \; \underline{\text{is equivalent to}} \; (not\ A) \textit{ or } (not\ B).$$

As before, verification using Venn diagrams is left for the exercises. The identity is seen more clearly through an example.

Example 2.20 We return to the events S (ownership of stock) and B (ownership of bonds) in the previous example. De Morgan's laws state that

$$\sim(S \cup B) = \sim S \cap \sim B$$

and

$$\sim(S \cap B) = \sim S \cup \sim B.$$

In words, the first identity states that if you don't own stocks *or* bonds then you don't own stocks *and* you don't own bonds (and vice versa). The second identity states that if you don't own both stocks *and* bonds, then you don't own stocks *or* you don't own bonds (and vice versa). ☐

De Morgan's laws and the distributive laws are worth remembering. They enable us to simplify events which are stated verbally or in set notation. They will be useful in the counting and probability problems which follow.

2.5 Counting

Since many (not all) probability problems will be solved by counting outcomes, this section will develop a number of counting principles which will prove useful in solving probability problems.

2.5.1 Basic Rules

We will first illustrate the basic counting rules by example and then state the general rules. In counting, we will use the convenient notation

$$n(A) = \text{the number of elements in the set (or event) } A.$$